THEORETICAL NEUROSCIENCE I

Lecture 8: Simple LIF networks and cortical operating regime

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Content

1. Synapses and LIF neurons (recap)

2. Spike train statistics

3. Coupled LIF neurons

4. Regular and irregular firing modes

1 Synapses and LIF neurons (recap)

The dynamic equation for the membrane potential $V(t)$ of a LIF neuron is

$$\tau_m \frac{dV(t)}{dt} = -[V(t) - E_L] + r_m I_e(t)$$

Resting potential E_L, electrode current $I_e(t)$ (if any), membrane time constant τ_m, and threshold and reset potentials V_{th} and V_{reset}:

$$\tau_m = 20\,ms, \quad E_L = -70\,mV, \quad V_{th} = -54\,mV, \quad V_{reset} = -80\,mV$$

Multiple synapses

With multiple input synapses, the dynamic equation becomes

$$\tau_m \frac{dV(t)}{dt} = -[V(t) - E_L] - r_m\, g_s \sum_k P_k(t)\, [V(t) - E_s]$$

The synaptic time courses $P_k(t)$ may be approximated with a suitable function, for example

$$P_s(t) = \left[{}^{\Delta t}/_{\tau_s} \exp\left(-{}^{\Delta t}/_{\tau_s}\right)\right]_+$$

where τ_s is a time-constant and $\Delta t = [t - t_i]_+$ is the (positive) time since the last presynaptic spike t_i.

Four steps from presynaptic to postsynaptic spikes

1. Presynaptic spike t_i triggers activation $P_s(t)$ for $t \geq t_i$.

2. Synaptic conductance contributes $V_{PSP}(t)$ to dynamics.

3. Equilibrium potential $V_\infty(t)$ is modulated.

4. Membrane potential $V_m(t)$ follows equilibrium potential with delay.

2 Spike train statistics

Networks of neurons produce spike trains. Before simulating networks, we collect some statistical tools that are useful for describing spike trains.

A *spike train* is a series of spikes in time

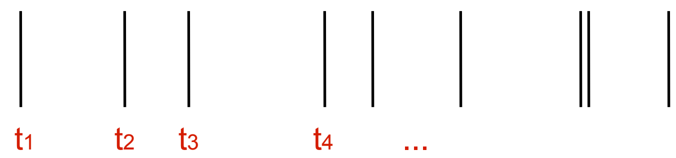

Figure 1: Spike train.

$$\{t_1, t_2, t_3, \ldots, t_N\} \qquad\qquad t_i < t_{i+1}$$

Spike trains are irregular and best described in statistical terms. They are many different kinds of irregularity!

Spike rate and spike interval

Important first-order measures are average rate $\lambda\,[Hz]$ and average interval $\langle t_{isi}\rangle\,[s]$

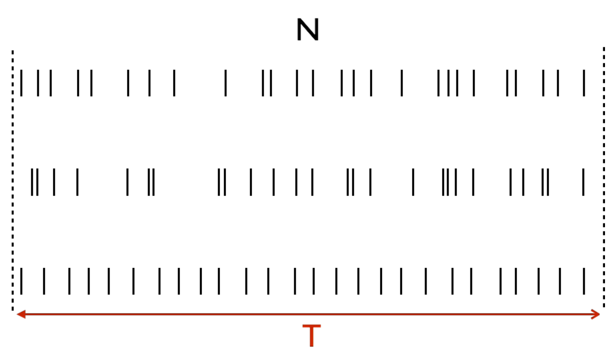

Figure 2: Different rates for spike trains.

$$\lambda = \frac{N}{T}, \qquad \langle t_{isi} \rangle = \frac{T}{N}, \qquad \lambda = \frac{1}{\langle t_{isi} \rangle}$$

Ensemble of intervals

To characterise irregularity, we can consider the *ensemble of intervals* $t_{isi} = t_{i+1} - t_i$

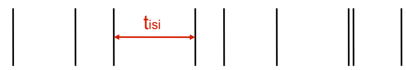

Figure 3: Spikes intervals

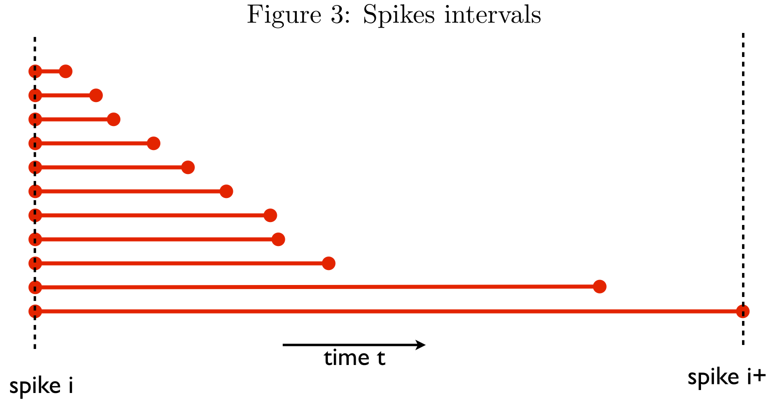

Figure 4: Spikes intervals represented schematically.

Interval count

Sort ensemble of intervals t_{isi} into 'bins' of size Δt:

$$[0, \Delta t] \qquad [\Delta t, 2\Delta t] \qquad \ldots \qquad [i\Delta t, (i+1)\Delta t] \qquad \ldots$$

In this way, we obtain *interval counts* n_i:

$$n_i \qquad \sum_i n_i = N$$

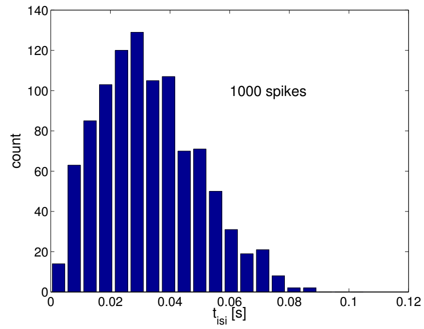

Figure 5: Interval count.

Interval fraction

To be independent of N, we can compute the *fraction* of intervals in each 'bin':

$$f_i = \frac{n_i}{\sum_i n_i} \qquad \sum_i f_i = 1$$

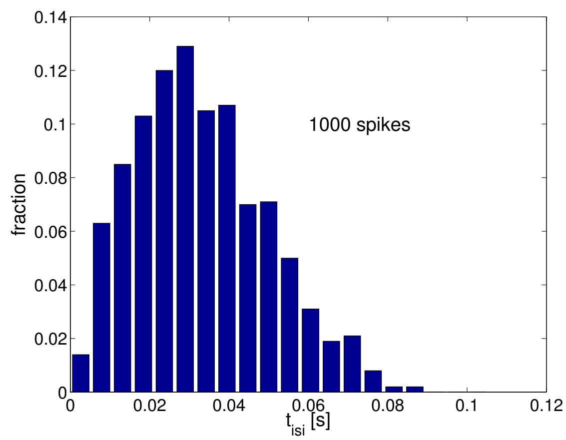

Figure 6: Interval fraction.

Interval density

To be independent of Δt, we can compute *interval density* (fraction per bin width):

$$p_i = \frac{n_i}{\Delta t \sum_i n_i} \qquad \sum_i p_i \, \Delta t = 1$$

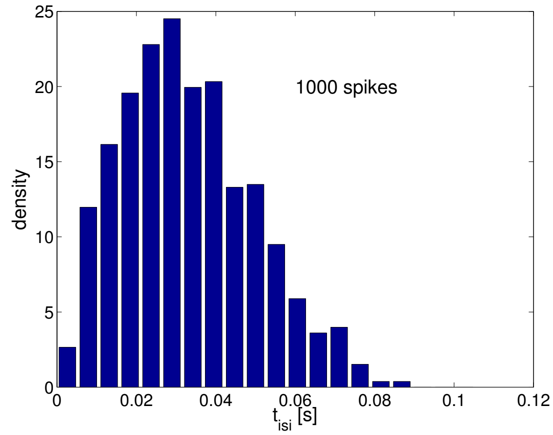

Figure 7: Interval density

Continuous density

In the limit of $\Delta t \rightarrow 0$ and $N \rightarrow \infty$, the interval density is continuous. The density takes units of $[s^{-1}]$ or $[Hz]$!

$$p(t) \quad \text{in } [s^{-1}] \qquad\qquad \int_0^\infty p(t)\,dt = 1$$

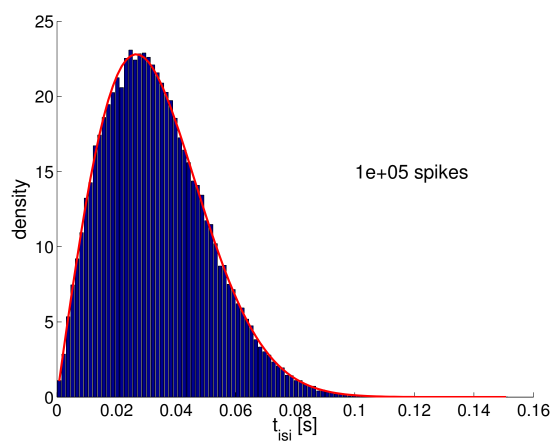

Figure 8: Continous density.

Coefficient of variation

To measure the irregularity of spike train, one (of several) possible measure is the *coefficient of variation* c_v of the intervals. It is computed from the mean interval $\langle t_{isi} \rangle$, the mean square interval $\langle t_{isi}^2 \rangle$, and the standard deviation of intervals σ_{isi}:

$$\langle t_{isi} \rangle = \frac{1}{N} \sum t_{isi} \qquad \langle t_{isi}^2 \rangle = \frac{1}{N} \sum t_{isi}^2$$

$$\sigma_{isi} = \sqrt{\langle t_{isi}^2 \rangle - \langle t_{isi} \rangle^2}$$

$$c_v \equiv \frac{\sqrt{\langle t_{isi}^2 \rangle - \langle t_{isi} \rangle^2}}{\langle t_{isi} \rangle} = \frac{\sigma_{isi}}{\langle t_{isi} \rangle}$$

Irregularity of example spike train

Our example spike train has a coefficient of variation of

$$c_v \approx 0.5$$

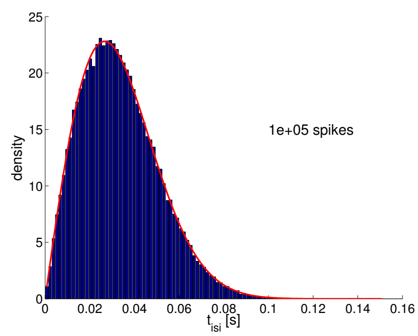

Figure 9: Irregularities in spike train.

Poisson spike trains

Events that occur independently of each other and with constant probability produce a particularly simple statistics (photon counts, radioactive decay, spikes, ...).

If we place $N >> 1$ points randomly and independently in an interval T, the resulting sequence approximates Poisson events with rate $\lambda = \frac{N}{T}$:

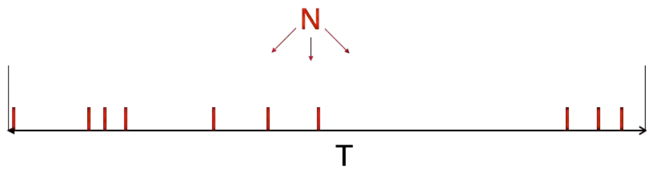

Figure 10: Poisson spike trains.

Poisson interval density

The interval density of Poisson events with rate λ is exponential

$$p(t_{isi}) = \begin{cases} \lambda\, e^{-\lambda\, t_{isi}} & t_{isi} \geq 0 \\ 0 & t_{isi} < 0 \end{cases}$$

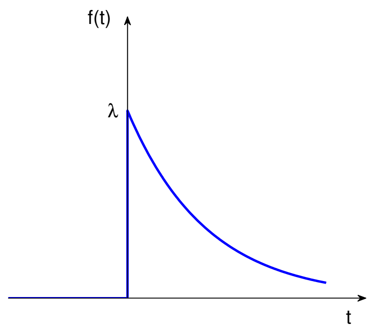

Figure 11: Poisson interval density

Proof (advanced)

Probability that *no* spike occurred in n intervals $\Delta t = t_{isi}/n$ and that *one* spike occurred in the next interval:

$$p(n\Delta t) = (1 - \lambda\Delta t)^n \cdot \lambda\Delta t$$

$$p(t_{isi}) = \lim_{n\to\infty} \frac{p(n\Delta t)}{\Delta t} = \lambda \lim_{n\to\infty} \left(1 - \frac{\lambda t_{isi}}{n}\right)^n = \lambda e^{-\lambda\, t_{isi}}$$

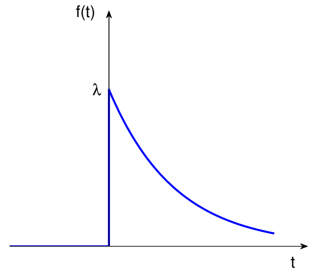

Figure 12: No spike in n intervals

Irregularity of Poisson spikes

For Poisson spikes, the coefficient of variation is

$$c_v \approx 1.0$$

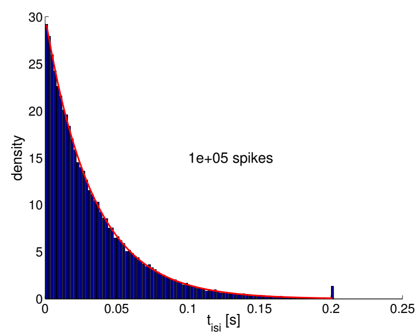

Figure 13: Irregularity in Poisson spikes

Summary spike statistics

- Spike trains are best described in statistical terms. Important measures are average rate $\lambda\,[Hz]$ and average interval $\langle t_{isi} \rangle\,[s]$
- A more detailed measure is the interval density $p(t_{isi})\,[Hz]$

$$p(t_{isi}) = \frac{P[t \leq t_{isi} \leq t + dt]}{dt}$$

- A measure of irregularity is the coefficient of variation

$$c_v \equiv \frac{\sqrt{\langle t_{isi}^2 \rangle - \langle t_{isi} \rangle^2}}{\langle t_{isi} \rangle} = \frac{\sigma_{isi}}{\mu_{isi}}$$

- Spike trains often resemble Poisson events (independent and uniformly likely). In this case, the interval density is exponential

$$p(t_{isi}) = \lambda\, e^{-\lambda\, t_{isi}}$$

3 Coupled LIF neurons

We now consider two IF neurons that are coupled with either excitatory or inhibitory synapses. How will the spiking of one neuron affect the other? What would you expect intuitively?
For the neurons, we assume $\tau_m = 20\,ms$, $E_L = -70\,mV$, $V_{th} = -54\,mV$, $V_{reset} = -80\,mV$.

Figure 14: Excitatory and inhibitory coupling

To make our mini-network more realistic, we simulate the synaptic influence of other neurons by adding a small but randomly fluctuating electrode current.

Excitatory or inhibitory synapses For the synapses, we assume the following reversal potentials and maximal conductances

$$E_{ex} = 0\,mV \qquad E_{in} = -80\,mV \qquad r_m\,g_{ex} = 0.025 \qquad r_m\,g_{in} = 0.1$$

as well as a moderately fast time-course of synaptic currents

$$P_{ex}(t) = \frac{\Delta t}{\tau_s}\,\exp\left(1 - \frac{\Delta t}{\tau_s}\right) \qquad\qquad P_{in}(t) = \frac{\Delta t}{\tau_s}\,\exp\left(1 - \frac{\Delta t}{\tau_s}\right)$$

where $\Delta t = [t - t_i]_+$ is the (positive) time since the last spike and $\tau_s = 10\,ms$.

Membrane voltage

In the range of $V_{reset} \leq V \leq V_{th}$, the dynamic equations for the membrane voltages are

$$\tau_m\,\frac{dV_1}{dt} = -V_1 + E_L - r_m\,g_{ex}\,P_{ex}^{(2)}(t)\ [V_1 - E_{ex}] + noise$$

$$\tau_m \frac{dV_2}{dt} = -V_2 + E_L - r_m\, g_{ex}\, P_{ex}^{(1)}(t)\ [V_2 - E_{ex}] + noise$$

$P_{ex}^{(1)}$ reflects synaptic transmission $1 \rightarrow 2$, and $P_{ex}^{(2)}$ transmission from $2 \rightarrow 1$.

Mutual excitation

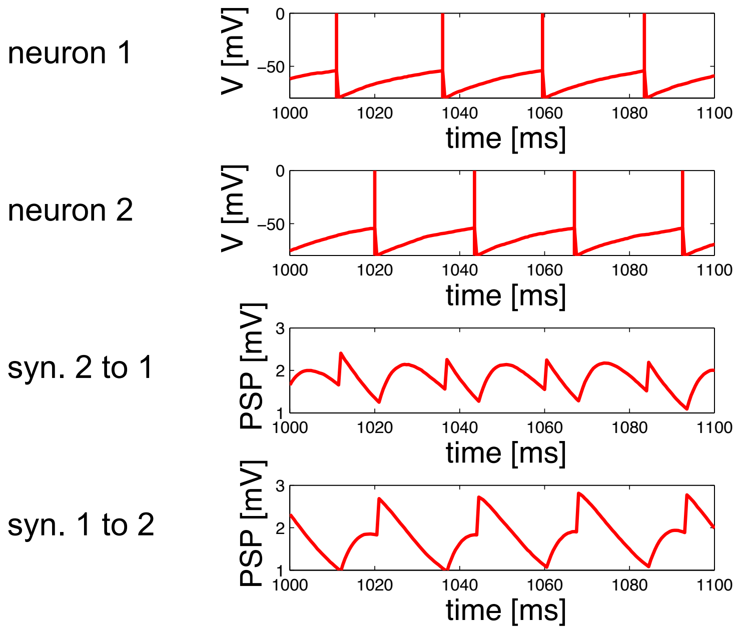

Figure 15: Mutual excitation

Mutual inhibition

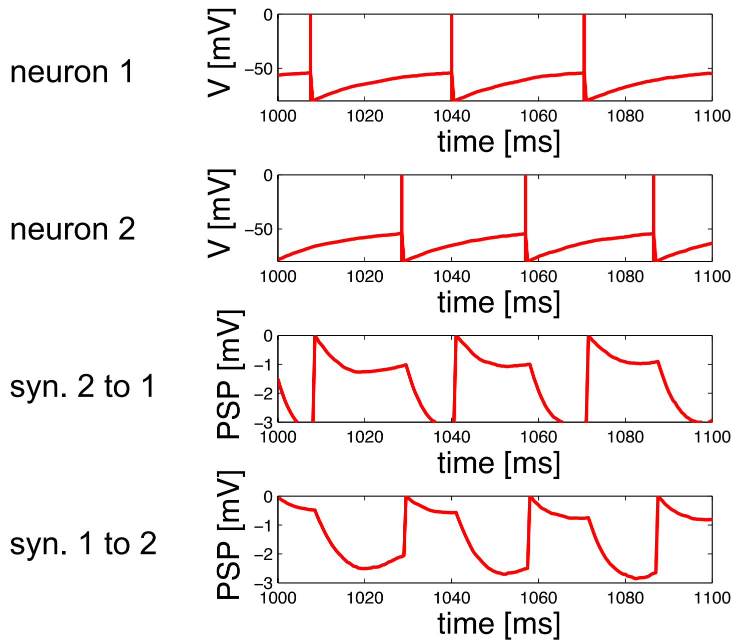

Figure 16: Mutual inhibition

Response function

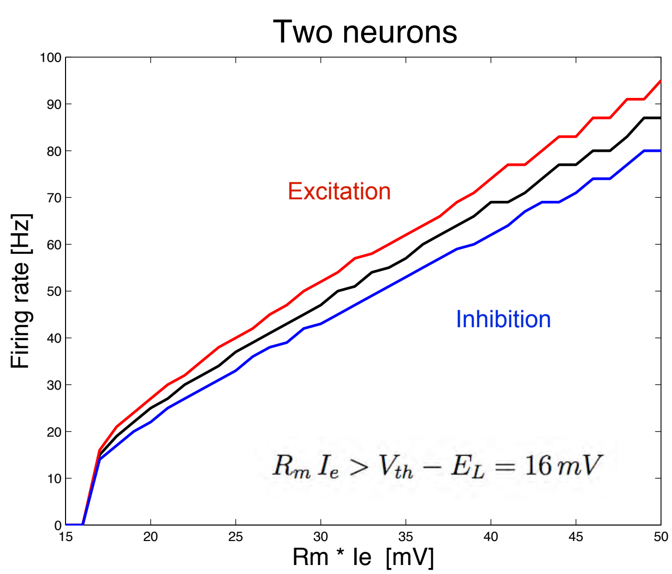

Figure 17: Response function of two neurons

Synchronous and alternating firing modes

When the synaptic time constant is sufficiently large, excitatory connections between two neurons can produce alternate firing, while inhibitory neurons produce synchronous firing.

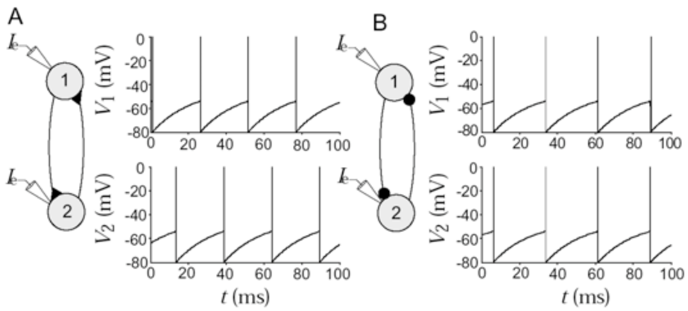

Figure 18: Synchronous and alternate firing. [1]

Mutual excitation can cause alternate firing.

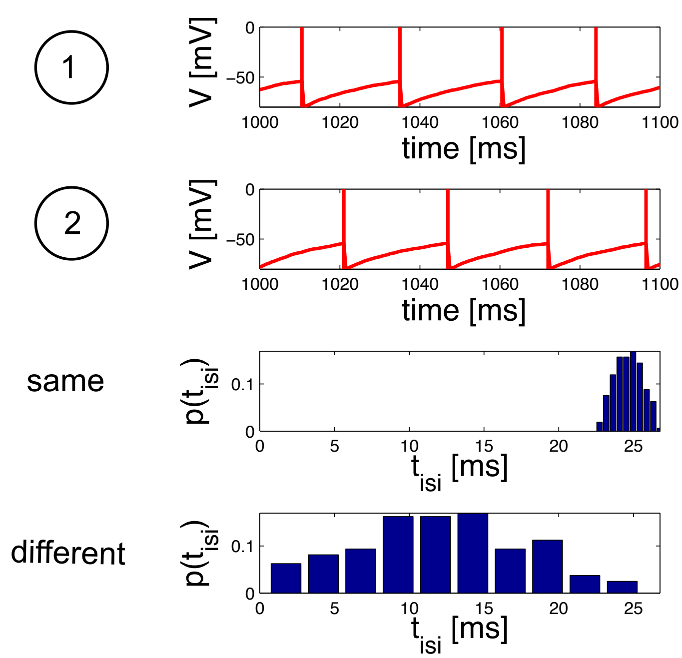

Figure 19: Alternate firing

Mutual inhibition can cause simultaneous firing.

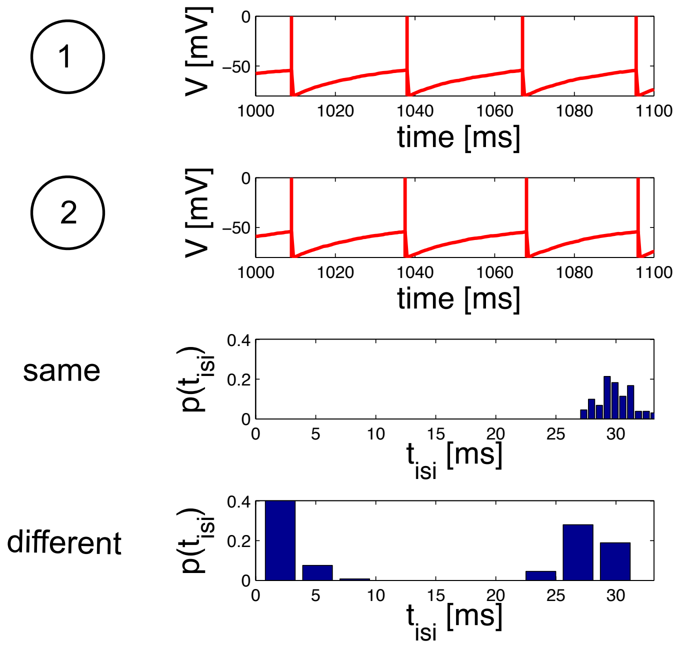

Figure 20: Simultaneous firing

Five coupled neurons

Figure 21: Excitatory and inhibitory coupling.

Response function

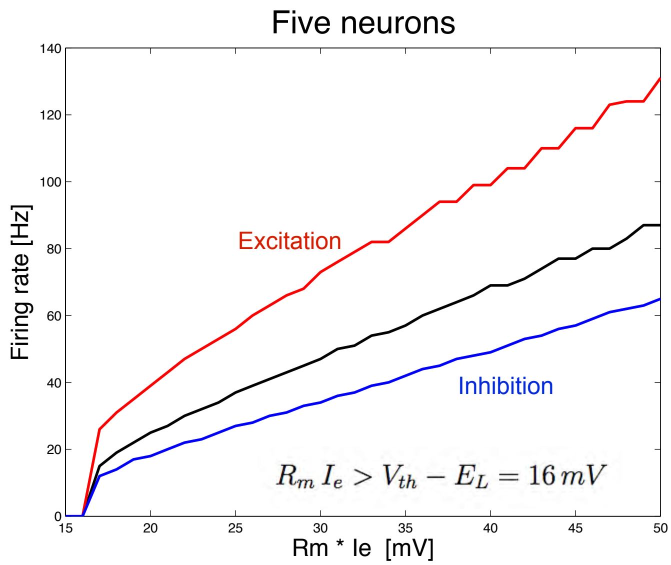

Figure 22: Response function

Mutual excitation

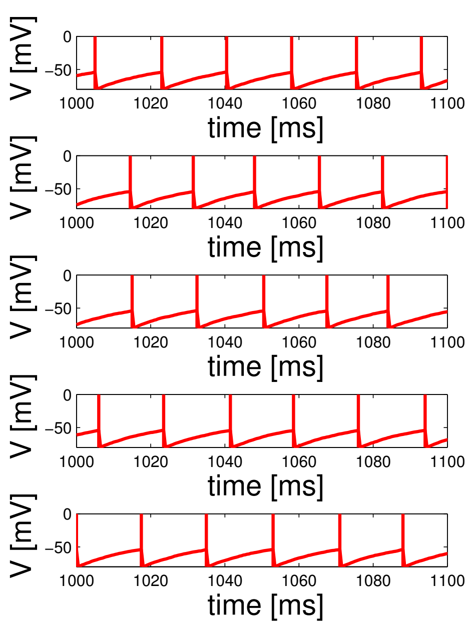

Figure 23: Mutual excitation

Mutual excitation tends to *desynchronize* spiking.

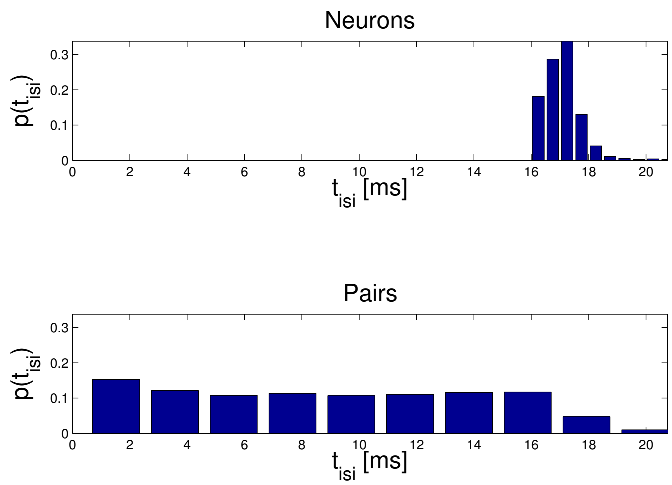

Figure 24: Desynchronize spiking

Mutual inhibition

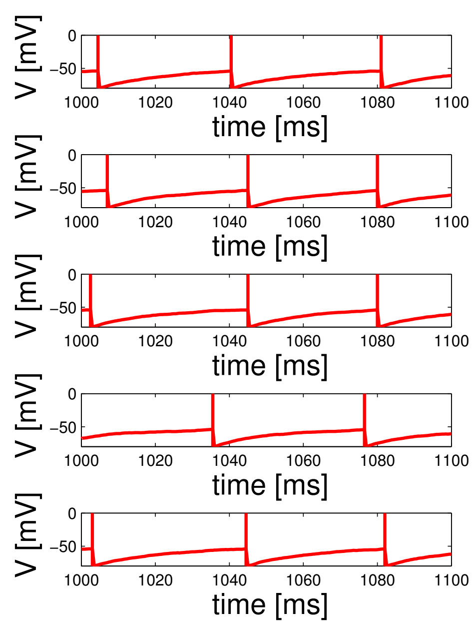

Figure 25: Mutual inhibition

Mutual inhibition tends to *synchronize* spiking.

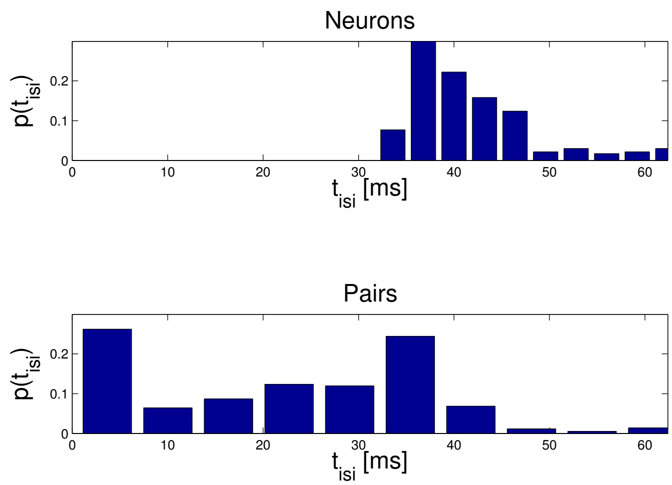

Figure 26: Resynchronization

Coupled LIF neurons

- Coupling LIF neurons with reciprocal synapses has several effects.
- Excitatory coupling *increases* and inhibitory coupling *decreases* the gain of the response function.

- Excitatory coupling tends to *desynchronize* spikes across the population.
- Inhibitory coupling tends to *synchronize* spikes across the population.
- Post hoc explanation: spike probability peaks when excitation is maximally effective and/or inhibition is minimally effective.

- **Collective behavior of even the simplest networks shows complex 'emergent' properties.**

4 Regular and irregular firing models

LIF neuron models are used for studying the summation of large numbers of synaptic inputs. Such models predict two distinct modes of operation, depending on the balance of excitation and inhibition:

(1) When inputs raise the average potential above the firing threshold, spiking is *regular* ($C_v << 1$) and governed by the membrane time constant (relaxation from V_{reset}).

(2) When the average potential remains below threshold, spiking reflects random fluctuations in the potential and is *irregular* ($C_v \approx 1$)
LIF neuron with many synaptic inputs

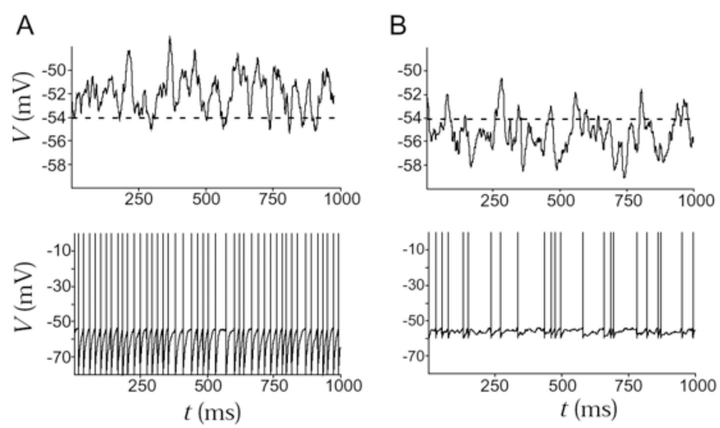

Figure 27: Postsynaptic membrane potential without (top) and with (bottom) spiking mechanism. [2]

Regular (left) and irregular (right) firing modes.
Presynpatic spike trains
Poisson spike trains with average rate 100 Hz.

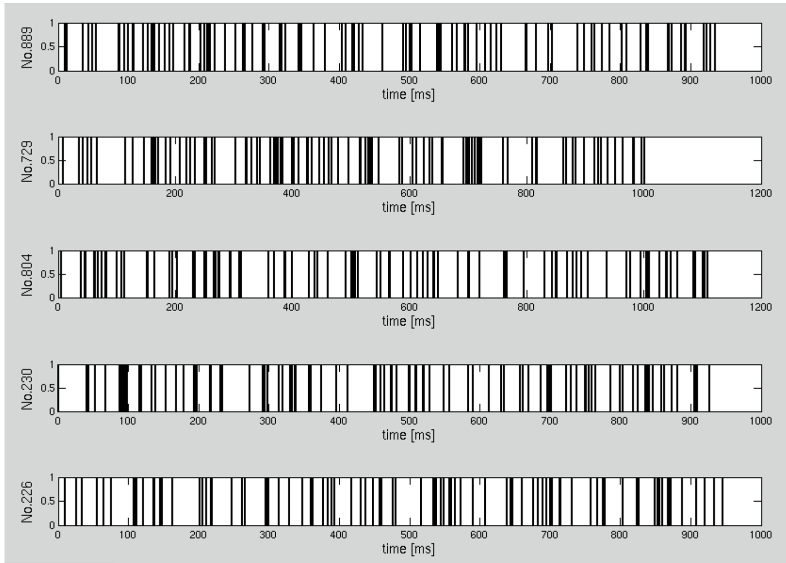

Figure 28: Presynaptic spike trains.

Cumulative effect of EPSPs and IPSPs

$$P_{ex} \approx g_{ex}\, r_m\, (V_{th} - E_{ex}) \sum_k P_{ex}^k(t), \qquad P_{in} \approx g_{in}\, r_m\, (V_{th} - E_{in}) \sum_k P_{in}^k(t)$$

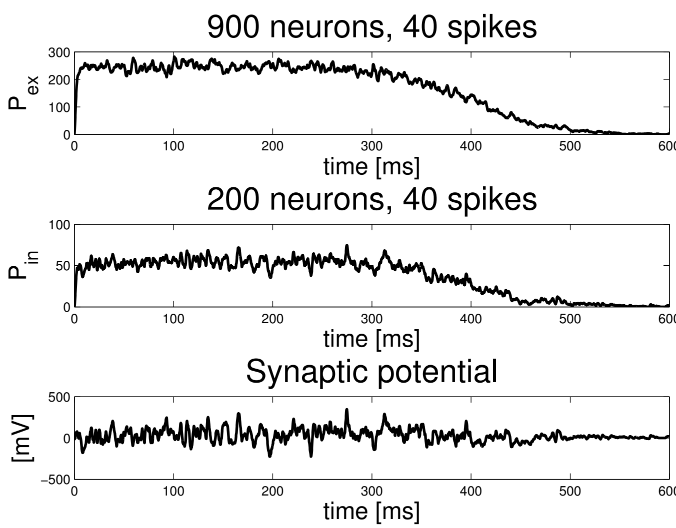

Figure 29: Cumulative effect of EPSPs and IPSPs.

Slow irregular firing mode

Result for 950 excitatory inputs ($g_{ex}\, r_m = 0.05$) and 200 inhibitory inputs ($g_{in}\, r_m = 0.5$), with $V_{reset} = -60\, mV$.

$$\text{rate} \quad \nu = 50\, Hz \qquad\qquad C_v = 0.82$$

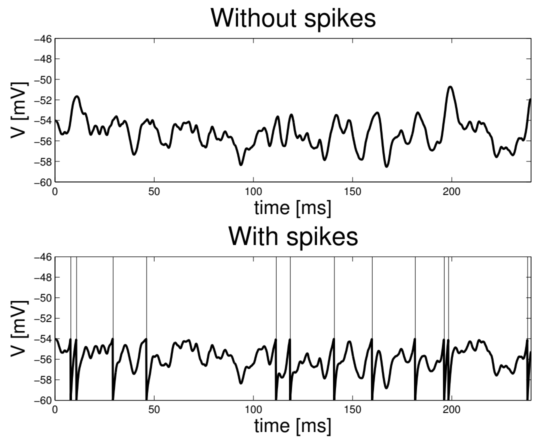

Figure 30: Slow irregular firing mode.

Rapid regular firing mode

Result for 1100 excitatory inputs ($g_{ex}\, r_m = 0.05$) and 200 inhibitory inputs ($g_{in}\, r_m = 0.5$), with $V_{reset} = -80\, mV$.

$$\text{rate} \quad \nu = 280\, Hz \qquad\qquad C_v = 0.48$$

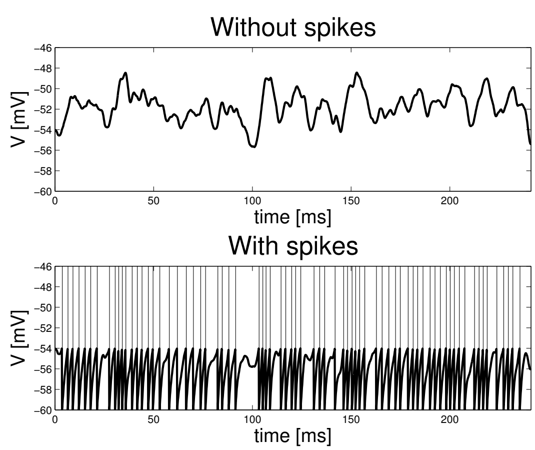

Figure 31: Rapid regular firing mode.

Mean and variance of membrane potential
Regular firing mode:

In the absence or spikes, mean membrane potential is *above* threshold. Spike timing is governed by *recovery* of membrane potential from rest. This results in more *regular* inter-spike intervals.

Irregular firing mode:

In the absence of spikes, mean membrane potential is *below* threshold. Spike timing is governed by *fluctuations* of membrane potential over and above mean. This results in more *irregular* inter-spike intervals.

Theory of membrane potential fluctuations

Consider synapses with individual postsynaptic potentials (PSP)

$$u(t) = w\,\epsilon(t) \qquad w = r_m\, g_s\,[V - E_s], \qquad \epsilon(t) = \exp\left(-\frac{t}{\tau_s}\right)$$

where w is synaptic weight and τ_s is the synaptic time-constant.

The sign of w determines whether the synapse is excitatory (EPSP) or inhibitory (IPSP).

What is the collective effect of many of such synapses? How do they change the membrane potential?

To answer this question, we assume that synaptic events are independent and equally likely to occur at any time (Poisson assumption).

Mean and variance of PSP (without proof!)

For Poisson PSP events with rate ν, weight w, and time-constant τ_s, the expected contribution to the *mean* membrane potential is

$$\mu_V = w\,\nu\,\tau_s$$

whereas the expected contribution to the *variance* is

$$\sigma_V^2 = \frac{1}{2} w^2 \, \nu \, \tau_s$$

Note that the *mean* contribution depends on w, but the *variance* contribution on w^2.

Balanced excitation and inhibition

For balanced excitation (EPSPs) and inhibition (IPSPs), with identical rates $\nu_+ = \nu_- = \nu$ and equal but opposite weights $w_+ = w$ and $w_- = -w$, the expected contributions are

$$\mu_V = w_+ \, \nu_+ \, \tau_s + w_- \, \nu_- \, \tau_s = w \, \nu \, \tau_s - w \, \nu \, \tau_s = 0$$

$$\sigma_V^2 = \frac{1}{2} w_+^2 \, \nu_+ \, \tau_s + \frac{1}{2} w_-^2 \, \nu_- \, \tau_s = w^2 \, \nu \, \tau_s$$

Balanced excitation and inhibition cancel in the mean, but add in the variance!

Fluctuation-driven output spikes

- The average of the membrane potential is set by the *difference* between excitatory and inhibitory input rates

$$\mu_V = w_+ \, \nu_+ \, \tau_+ - w_- \, \nu_- \, \tau_-$$

- The variance of the membrane potential is set by the *sum* of excitatory and inhibitory input rates

$$\sigma_V^2 = \frac{1}{2} \, w_+^2 \, \nu_+ \, \tau_+ + \frac{1}{2} \, w_-^2 \, \nu_- \, \tau_-$$

- The *difference* between excitatory and inhibitory input controls the firing mode (slow & irregular or fast & regular).

- In the slow & irregular mode, the *sum* of excitation and inhibition controls output spiking.

Summary regular and irregular firing modes

- Neurons receive thousands or tens of thousands of inputs. Typically, excitatory and inhibitory inputs are approximately balanced.
- LIF neuron models reveal the functional implications of balanced excitation and inhibition.
- There normal operating mode is $\langle V_m \rangle < V_{thresh}$, with slow, irregular firing.
- A pathological mode is $\langle V_m \rangle > V_{thresh}$, with fast, regular firing.
- Slow, irregular mode is controlled by *difference* of inputs (excitatory and inhibitory).
- Output firing is driven by fluctuations and controlled by *sum* of inputs (excitatory and inhibitory).
- Output firing is irregular and approximates Poisson statistics ($C_v \approx 1$).

5 Bibliography

1. Dayan & Abbott (2001), “Theoretical Neuroscience”, MIT Press, Ch. 1, fig. 1.14
2. Dayan & Abbott (2001), “Theoretical Neuroscience”, MIT Press, Ch. 1 fig. 1.18